

# BLACK HOLES IN HIGHER CURVATURE GRAVITY<sup>1</sup>

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## 1. Introduction

The idea that the Einstein action should be modified by the addition of interactions involving higher powers of the Riemann curvature tensor has a long history stretching back to the early days of general relativity. Such higher curvature theories originally appeared in proposals by Weyl and Eddington for a geometric unification of electromagnetism and gravity[1]. Much later, interest arose in higher derivative theories of gravity because they provided renormalizable quantum field theories[2, 3]. Unfortunately, the new massive spin-two excitations, which tame the ultraviolet divergences in such theories, result in the instability of the classical theory[4] and the loss of unitarity in the quantum theory[3, 5]. While higher curvature theories have thus proven inadequate as the foundation of quantum gravity, they still have a role to play within the modern paradigm of effective field theories[6].

Irrespective of the fundamental nature of quantum gravity, there should be a low energy effective action which describes the dynamics of a “background metric field” for sufficiently weak curvatures and sufficiently long distances. On general grounds, this effective gravity action will consist of the usual Einstein action plus a series of covariant, higher-dimension interactions, *i.e.*, higher curvature terms, and also higher derivative terms involving the “low-energy” matter fields. The appearance of such interactions can be seen, for example, in the renormalization of quantum field theory in curved space-time[7], or in the construction of low-energy effective

<sup>1</sup>To appear in: *Black Holes, Gravitational Radiation, and the Universe: Essays in Honor of C.V. Vishveshwara*, eds., C.V. Vishveshwara, B.R. Iyer, B. Bhawal

actions for string theory[8, 9]. In this context, the higher curvature interactions simply produce benign perturbative corrections to Einstein gravity coupled to conventional matter fields[10, 11].

Naive dimensional analysis would suggest that the coefficients of the higher dimension terms in such an effective Lagrangian should be dimensionless numbers of order unity times the appropriate power of the Planck length. Thus one might worry that all of the effects of the higher curvature terms would be the same order as those of quantum fluctuations, and so there would seem to be little point in studying modifications of “classical” black holes from higher dimension terms. One motivation for studying a classical higher curvature theory is that it is, of course, possible that the coefficients of some higher dimension terms are larger than what would be expected from simple dimensional analysis. Moreover, it is interesting to explore black holes in generalized gravity theories in order to discover which properties are peculiar to Einstein gravity, and which are robust features of all generally covariant theories of gravity. Further even within this framework, one may still discover clues as to the ultimate nature of the underlying theory.

With these introductory remarks, we will go on to discuss some of the recent investigations of black holes in higher curvature theories of gravity. The remainder of this chapter is organized as follows: In sect. 2, we briefly review various black hole solutions which appear in the literature. In sect. 3, we focus on black hole thermodynamics[12] in the context of higher curvature gravity. Finally in sect. 4, we provide a brief discussion and indicate some of the open questions.

At this point, let us add that many of the recent candidates for a theory of quantum gravity, especially those which attempt to unify gravity with other interactions, are theories for which the space-time dimension is greater than four. Thus much of the following discussion refers to black holes in higher dimensional space-times[13]. While this idea may seem unusual and/or unappealing to some readers, it is certainly one familiar to our esteemed colleague, C.V. Vishveshwara[14, 15], to whom this volume is dedicated. Throughout, we also employ the conventions of ref. [16] in any formulae.

## 2. Black Hole Solutions

When faced with the task of finding solutions of higher curvature gravity (or any higher derivative theory), one must realize that it is incorrect to assume that adding higher derivative correction terms with “small” coefficients will *only* produce small modifications of the solutions of the unperturbed theory[11]. Any higher curvature theory will, in fact, contain whole classes

of new solutions unavailable to the classical Einstein theory. In particular, the set of maximally symmetric vacuum solutions will now include some number of (anti-)deSitter vacua, as well as flat space[17, 18]. A common feature of these new solutions is that they are not analytic in the coefficients of the higher derivative interactions, and hence in the context of an effective field theory, they should be regarded as unphysical. Systematic procedures have been developed to exclude these spurious solutions[11, 19]. However, in the context of effective field theory where the domain of validity of the equations of motion is expected to be limited, it suffices to treat the higher curvature contributions perturbatively.

For example, let us consider the field equations:  $R_{ab} = \alpha H_{ab}$  where  $R_{ab}$  is the Ricci tensor,  $H_{ab}$  is some higher derivative contribution, and  $\alpha$  is the (dimensionful) coefficient of the higher curvature term in the action. Now making the obvious expansion of the solution,  $g_{ab} = g_{ab}^0 + \alpha g_{ab}^1 + \dots$ , one must solve

$$R_{ab}(g^0) = 0 \quad (1)$$

$$(\Delta[g^0] g^1)_{ab} = H_{ab}(g^0) \quad (2)$$

where  $\Delta[g^0]$  is the second order operator obtained from linearizing the Ricci tensor about the metric  $g_{ab}^0$ . Solving eq. (1) amounts to selecting a solution of the unperturbed Einstein theory. Solving eq. (2) is solving for a linearized fluctuation in that background with some source term. As well as imposing appropriate boundary conditions on  $g_{ab}^1$  (*e.g.*, that it preserve asymptotic flatness and regularity of the horizon), one would (usually) choose it to preserve the symmetries of the original solution (*e.g.*, spherical symmetry, stationarity). While the above discussion was phrased in terms of purely gravitational solutions, it would be straightforward to include matter fields, as one might in considering charged black holes. Further this approach extends in an obvious way to developing the perturbation expansion to higher orders, which may involve including additional higher order interactions.

In this expansion, the coupling constant  $\alpha$  has the dimension of length to some (positive) power  $n$ , *e.g.*,  $n = 2$  if  $H_{ab}$  arises from a curvature-squared interaction. The true dimensionless expansion parameter in the above analysis is then  $\alpha/L^n$  where  $L$  is the local curvature scale. Hence this expansion will always break down in the black hole interior near the singularity, but it will be reliable in the exterior region of a large black hole where the curvatures are small. Thus within this context, one can expect that for large black holes the asymptotic regions feel only minor corrections due to the higher curvature terms. Near the singularity, the higher curvature contributions become strong, but this implies that one has left the realm in which the effective action is to be trusted. It would

seem that at this point one must come to grips with the full underlying fundamental theory. However, one might attempt to make models of high curvature behavior[20, 21] to guide our intuition.

Within this framework, it is interesting to consider the propagation of metric disturbances in the black hole background. One would organize the disturbance with the same  $\alpha$  expansion as above:  $g_{ab} = (g^B + h)_{ab} = (g^0 + h^0)_{ab} + \alpha(g^1 + h^1)_{ab}$  where  $g^B_{ab} = g^0_{ab} + \alpha g^1_{ab}$  is the background metric satisfying eqs. (1) and (2), above. The disturbance then satisfies

$$\left( \Delta[g^0] h^0 \right)_{ab} = 0 \quad (3)$$

$$\left( \Delta[g^0] h^1 \right)_{ab} = J_{ab}(g^0, h^0) \quad (4)$$

where  $J_{ab}(g^0, h^0)$  is the linearization of  $H_{ab}(g^0 + h^0)$ . Note that within this scheme, the propagation of metric disturbances and hence the causal structure are completely determined by the original background metric  $g^0_{ab}$ . Hence we are guaranteed that the black hole really remains a black hole, and the event horizon remains an event horizon.

At some level, the conclusions of the previous paragraph may actually seem somewhat surprising. For instance, one would conclude that rather than following null geodesics in the perturbed background  $g^B_{ab} = g^0_{ab} + \alpha g^1_{ab}$ , “high frequency” gravity waves still follow null geodesics of the original metric  $g^0_{ab}$ . In either case, one would expect these conclusions to apply in an approximation:  $L^n \gg \lambda^n \gg \alpha$  where  $\lambda$  is the wavelength of the disturbance. The first inequality justifies a geometric optics approximation[16], whereas the second is required for the reliability of the effective action. Now in examining the propagation of wavefronts in curved space, one expects curvature corrections to appear at the order  $\lambda/L$ , whereas the modifications to the metric due to the higher curvature interactions are of the order  $\alpha/L^n$ . Hence given the above inequalities, one has  $\lambda/L \gg (\lambda/L)^n \gg \alpha/L^n$ , and so the discrepancy between using either  $g^B_{ab}$  or  $g^0_{ab}$  to define null geodesics is certainly a subleading correction in determining the propagation of gravitational disturbances. However, using  $g^0_{ab}$  seems more consistent in the application of perturbation theory[22, 23].

Eqs. (3) and (4) are also relevant in addressing the important question of the (linearized) stability of the horizon[26] — a topic to which Vishu has made seminal contributions[27]. Given that the original Einstein theory was stable, one knows that there are no runaway solutions to eq. (3). Eq. (4) simply extends the latter equation by the addition of a regular source term. Hence it is also clear that  $h^1_{ab}$  has no runaway solutions by the application of the original stability analysis. Thus one can immediately deduce that the black hole solution remains stable within this perturbative framework.

A perturbative approach has been applied in examining modifications of the four-dimensional Schwarzschild black hole within the context of renormalized Einstein gravity[28, 29]. In this case, curvature-squared interactions do not produce any modifications for solutions of the four-dimensional vacuum Einstein equations. Hence this analysis considered Einstein gravity perturbed by the addition of terms involving three Riemann curvatures. The main observation resulting from this analysis[28] was that the relations between the black hole's mass and its thermodynamic parameters[12] are modified. In particular, the black hole entropy was no longer proportional to the area of the horizon.

Various perturbative analyses have also been made to study black holes in string theory[30]–[33]. These include considering the effects of curvature-squared terms on spherically symmetric black holes in arbitrary dimensions[31], and on four-dimensional black holes with angular momentum[32] or with charge[33].<sup>2</sup> In certain supersymmetric string theories, the leading higher curvature interaction can be shown to contain four curvatures[35]. The effect of these terms on spherically symmetric black holes in arbitrary dimensions has also been studied[30]. Apart from modifications to the usual thermodynamic properties of the black holes, one of the remarkable observations here was the fact that the higher curvature terms induce various new forms of long-range scalar field hair on the black holes. However, this new hair may be regarded as secondary[36], in that it is completely determined by the black hole's primary hair, *e.g.*, the mass and charge. In other words, there are no new constants of integration, and hence these solutions do not violate the spirit of the no-hair theorems[37, 38]. Essentially, the new hair arises because the scalar fields have non-minimal couplings to the higher curvature terms.

Motivated originally by string theory, a great deal of attention has been focussed on Lovelock gravity[39]. The latter is defined by a Lagrangian which is the sum of dimensionally extended Euler densities. In four dimensions, all of the higher curvature terms are total derivatives, and hence the theory reduces to Einstein gravity. However, in higher dimensions, the new interactions do make nontrivial contributions. A distinguishing feature of these Lagrangians is that the resulting equations of motion contain no more than second derivatives in time[39]. As quantum theories then, they are free of ghosts when expanding about flat space[40], and so they evade the problem of unitarity loss — however, they remain nonrenormalizable. Exact spherically symmetric solutions were first found for the curvature-squared or Gauss-Bonnet theory[18]. These results were quickly extended to arbitrary Lovelock theories[41]–[43], as well as charged black holes[44].

<sup>2</sup>Note that in string theory, charged black holes typically differ from the Reissner-Nordstrom geometry as a result of nonminimal couplings between the matter fields[34].

These solutions displayed a rich structure of multiple horizons, and unusual thermodynamic properties[42]. For example, certain (uncharged) solutions could be found with vanishing Hawking temperature[45]. The topic of Lovelock black holes is, in fact, another area upon which Vishnu's research has touched[15].

While in four dimensions, the Lovelock action yields only Einstein gravity, one can also consider the Kaluza-Klein compactification[46] of a higher dimensional Lovelock theory down to four dimensions. The resulting theory consists of Einstein gravity coupled to various scalar and vector fields with nonminimal higher-derivative interactions[47]. In this case, there are four-dimensional black holes which carry secondary scalar hair[48].

More recently researchers[49, 50], interested in whether the secondary scalar hair found in ref. [31]–[33] survived beyond perturbation theory, investigated exact solutions of a (four-dimensional) dilatonic Gauss-Bonnet theory. In the latter, the curvature-squared interaction is modified by the addition of a nonminimal scalar coupling. The full equations of this theory are difficult enough that they could not be solved analytically. However, analytic arguments and numerical evidence indicates that the modified black holes do carry secondary scalar hair[49]. These results were also extended to black holes carrying charge[50].

Some work[51, 52] has also been done on black holes in theories where the Lagrangian density takes the form  $\sqrt{-g}f(R)$  where  $f(R)$  is a polynomial in the Ricci scalar. These models are amenable to analysis because they can be mapped to a theory of Einstein gravity and a minimally coupled scalar with an unusual potential[51, 53]. Note that if other matter fields are included then the latter develop unusual nonminimal couplings in the Einstein-scalar theory. In the case  $f = R + a_2R^2$ , a uniqueness theorem was proven for certain classes of matter fields in four dimensions[51]. Provided<sup>3</sup> that  $a_2 > 0$ , no new hair can arise and so the only black hole solutions are identical to those of Einstein gravity. Ref. [52] extended this work to spherically symmetric solutions for general polynomial actions in arbitrary dimensions. One finds that, with the same restriction on the quadratic term and irrespective of the remaining terms in the polynomial  $f$ , the only asymptotically flat black holes are still the Schwarzschild solutions.

The latter investigations of the Lovelock, dilatonic Gauss-Bonnet and polynomial-in- $R$  theories all go beyond the perturbative approach originally described and consider exact solutions of the full higher derivative equations of motion. Certainly amongst the exact solutions, one will find some which lend themselves to a Taylor expansion in the higher curvature coupling constants. These solutions could then be considered the result of carrying

<sup>3</sup>Note that this condition is also required for the stability of the theory[54, 55].

out the perturbation expansion to infinite order. To be of interest in the effective field theory framework though, one would have to know that there are no additional higher curvature interactions at higher orders. In general, this seems an unlikely scenario, and in string theory, certainly one that does not apply[56]<sup>4</sup> Although the physical motivation may not be strong, one can still set out to study these theories and their solutions as a mathematical problem in its own right. In this respect, a common advantageous feature, which the three theories discussed here seem to share in common, is the absence of negative energy ghosts[40, 54, 55, 58] — at least with certain restrictions on the coupling constants.

From this point of view, one should readdress the important question of the stability of the horizon for black hole solutions in these theories. Of course, the linearized stability for the full higher derivative equations is an extremely difficult problem<sup>5</sup>, however, some limited results have been achieved. Ref. [55] shows that the four-dimensional Schwarzschild black hole is stable in a general fourth order gravity theory. There also some limited results indicating stability of spherically symmetric four-dimensional black holes in the dilatonic Gauss-Bonnet theory[60].

A more fundamental question which should also be considered is the actual causal structure of the “black hole” solutions. It would seem that for the three theories of interest here, the Lovelock, dilatonic Gauss-Bonnet and polynomial-in- $R$  gravities, that the full equations are (or can be mapped to) second-order hyperbolic systems. However, the characteristic surfaces of these equations need not coincide with null cones in the background space-time, opening the possibility that gravitational disturbances could propagate “faster than light.” Note that such acausal behavior would result immediately if the theory in question had negative energy ghosts[61], which, although a feature of generic higher curvature theories, is not the case for these three theories. The possibility that gravitational disturbances could escape a “black hole” would have profound consequences for these theories. In the case of the Lovelock theories, investigations have been made of the modified characteristics[62], and in this case, a preliminary study[63] of radial wavefronts in static black holes suggests that disturbances can not escape the horizon. The same result would necessarily seem to apply for the polynomial-in- $R$  theories, which are essentially mapped to Einstein gravity (with a conformally related metric) coupled to a massive scalar field[53]. We stress, however, that we view the motivation in studying these theories as more mathematical than physical.

<sup>4</sup>In certain cases, however, one can use supersymmetry to argue that special solutions are exact to all orders despite the higher curvature corrections to the action[57].

<sup>5</sup>Note that going to higher dimensions introduces complications by itself[59].

### 3. Black Hole Thermodynamics

Black hole thermodynamics[12] is certainly one of the most remarkable features of the area of physics to which this volume is devoted. It produces a confluence of ideas from thermodynamics, quantum field theory and general relativity. Much of the interest in black hole thermodynamics comes from the hope that it will provide some insight into the nature of quantum gravity. While originally developed in the context of Einstein gravity, it is easy to see that much of the framework should extend to higher curvature theories, as well. Hawking's celebrated result[64] that a black hole emits thermal radiation with a temperature proportional to its surface gravity,  $\kappa$ :

$$k_B T = \frac{\hbar \kappa}{2\pi c} \quad (5)$$

is a prediction of quantum field theory in space-time containing a horizon[7], independent of the details of the dynamics of the gravity theory. Alternatively, this result will follow from the evaluation of the of the black hole partition function using the Euclidean path integral method[65]. Applying the latter approach in higher curvature gravity, it is also clear that one can derive a version of the First Law of black hole mechanics

$$\frac{\kappa}{2\pi c} \delta \mathcal{S} = c^2 \delta M - \Omega^{(\alpha)} \delta J_{(\alpha)} \quad (6)$$

where  $M$ ,  $J_{(\alpha)}$  and  $\Omega^{(\alpha)}$  are the black hole mass, canonical angular momentum, and the angular velocity of the horizon[66]. Given the identification of the black hole temperature with the surface gravity in eq. (5), the black hole entropy is naturally identified as  $S = (k_B/\hbar)\mathcal{S}$ . In the context of Einstein gravity, this gives the famous Bekenstein-Hawking entropy[64, 68]:

$$S_{BH} = \frac{k_B c^3}{\hbar G} \frac{A_H}{4} . \quad (7)$$

where  $A_H$  is the area of the event horizon[67].

Many of the early investigations[28, 31, 30, 42, 44, 45] which examined particular black hole solutions in higher curvature theories noted that eq. (7) no longer applied — see ref. [69] for a review. Important conceptual progress was made in ref. [70], where it was realized that  $\mathcal{S}$  should take the form of a geometric expression evaluated at the event horizon. This result was explicitly demonstrated there for the special case of Lovelock gravity, by extending a Hamiltonian derivation of the First Law[71]. Various other techniques were then developed[72]–[77] to expand this result to other higher curvature theories. In particular though, Wald[72] developed an elegant new derivation of the First Law which applies for any diffeomorphism invariant theory — see below. His derivation makes clear that in

eq. (6),  $\mathcal{S}$  may always be expressed as a *local* geometric density integrated over a space-like cross-section of the horizon.

### 3.1. BLACK HOLE ENTROPY AS NOETHER CHARGE

Here, we will provide a brief introduction to Wald's derivation of the First Law. The interested reader is referred to Refs. [72, 74, 75, 78] for a complete description. In the following, we also adopt the standard convention of setting  $\hbar = c = k_B = 1$ .

An essential element of Wald's approach is the Noether current associated with diffeomorphisms[79]. Let  $L$  be a Lagrangian built out of some set of dynamical fields, including the metric, collectively denoted as  $\psi$ . Under a general field variation  $\delta\psi$ , the Lagrangian varies as

$$\delta(\sqrt{-g}L) = \sqrt{-g}E \cdot \delta\psi + \sqrt{-g}\nabla_a\theta^a(\delta\psi) , \quad (8)$$

where “.” denotes a summation over the dynamical fields including contractions of tensor indices. Then the equations of motion are  $E = 0$ . With symmetry variations for which  $\delta(\sqrt{-g}L) = 0$ ,  $\theta^a$  is the Noether current which is conserved when the equations of motion are satisfied — *i.e.*,  $\nabla_a\theta^a(\delta\psi) = 0$  when  $E = 0$ . For diffeomorphisms, where the field variations are given by the Lie derivative  $\delta\psi = \mathcal{L}_\xi\psi$ , the variation of a covariant Lagrangian is a total derivative,  $\delta(\sqrt{-g}L) = \mathcal{L}_\xi(\sqrt{-g}L) = \sqrt{-g}\nabla_a(\xi^a L)$ . Thus one constructs an improved Noether current,

$$J^a = \theta^a(\mathcal{L}_\xi\psi) - \xi^a L ,$$

which satisfies  $\nabla_a J^a = 0$  when  $E = 0$ .

A fact[80], which may not be well-appreciated, is that for *any* local symmetry, there exists a globally-defined Noether potential  $Q^{ab}$ , satisfying  $J^a = \nabla_b Q^{ab}$  where  $Q^{ab} = -Q^{ba}$ .  $Q^{ab}$  is a local function of the dynamical fields and a linear function of the symmetry parameter (*i.e.*,  $\xi^a$  in the present case). Of course, this equation for  $J^a$  is valid up to terms which vanish when the equations of motion are satisfied. Given this expression for  $J^a$ , it follows that the Noether charge contained in a spatial volume  $\Sigma$  can be expressed as a boundary integral  $\oint_{\partial\Sigma} d^{D-2}x \sqrt{h} \epsilon_{ab} Q^{ab}$ , where  $h_{ab}$  and  $\epsilon_{ab}$  are the induced metric and binormal form on the boundary  $\partial\Sigma$ .

Another key concept that enters in Wald's construction is that of a Killing horizon. Given a Killing vector field which generates an invariance for a particular solution — *i.e.*,  $\mathcal{L}_\xi\psi = 0$  for all fields — a Killing horizon is a null hypersurface whose null generators are orbits of the Killing vector. If the horizon generators are geodesically complete to the past (and if the surface gravity is nonvanishing), then the Killing horizon contains a space-like cross-section  $B$ , the *bifurcation surface*, on which the Killing field  $\chi^a$

vanishes[81]. It can be shown that the event horizon of any black hole, which is static or is stationary with a certain  $t-\phi^{(\alpha)}$  orthogonality condition, must be a Killing horizon[82].<sup>6</sup> A stronger result holds in general relativity, where it can be proven that the event horizon for any stationary black hole is Killing[84].

The key to Wald's derivation of the First Law is the identity

$$\delta H = \delta \int_{\Sigma} dV_a J^a - \int_{\Sigma} dV_a \nabla_b (\xi^a \theta^b - \xi^b \theta^a), \quad (9)$$

where  $H$  is the Hamiltonian generating evolution along the vector field  $\xi^a$ , and  $\Sigma$  is a spatial hypersurface with volume element  $dV_a$ . This identity is satisfied for arbitrary variations of the fields away from any background solution. If the variation is to another solution, then one can replace  $J^a$  by  $\nabla_b Q^{ab}$ , so the variation of the Hamiltonian is given by surface integrals over the boundary  $\partial\Sigma$ . Further, if  $\xi^a$  is a Killing vector of the background solution, then  $\delta H = 0$ , and in this case, one obtains an identity relating the various surface integrals over  $\partial\Sigma$ .

Suppose that the background solution is chosen to be a stationary black hole with horizon-generating Killing field  $\chi^a \partial_a = \partial_t + \Omega^{(\alpha)} \partial_{\phi^{(\alpha)}}$ , and the hypersurface  $\Sigma$  is chosen to extend from asymptotic infinity down to the bifurcation surface where  $\chi^a$  vanishes. The surface integrals at infinity then yield precisely the mass and angular momentum variations,  $\delta M - \Omega^{(\alpha)} \delta J_{(\alpha)}$ , appearing in Eq. (6), while the surface integral at the bifurcation surface reduces to  $\delta \oint_B d^{D-2}x \sqrt{h} \epsilon_{ab} Q^{ab}(\chi)$ . Finally, it can be shown that the latter surface integral always has the form  $(\kappa/2\pi) \delta \mathcal{S}$ , where  $\kappa$  is the surface gravity of the background black hole, and  $\mathcal{S} = 2\pi \oint_B d^{D-2}x \sqrt{h} \epsilon_{ab} Q^{ab}(\tilde{\chi})$ , with  $\tilde{\chi}^a$ , the Killing vector scaled to have unit surface gravity.

By construction  $Q^{ab}$  involves the Killing field  $\tilde{\chi}^a$  and its derivatives. However, this dependence can be eliminated as follows[72]: Using Killing vector identities,  $Q^{ab}$  becomes a function of only  $\tilde{\chi}^a$  and the first derivative,  $\nabla_a \tilde{\chi}_b$ . At the bifurcation surface, though,  $\tilde{\chi}^a$  vanishes and  $\nabla_a \tilde{\chi}_b = \epsilon_{ab}$ , where  $\epsilon_{ab}$  is the binormal to the bifurcation surface. Thus, eliminating the term linear in  $\tilde{\chi}^a$  and replacing  $\nabla_a \tilde{\chi}_b$  by  $\epsilon_{ab}$  yields a completely geometric functional of the metric and the matter fields, which may be denoted  $\tilde{Q}^{ab}$ . One can show that the resulting expression,

$$\mathcal{S} = 2\pi \oint d^{D-2}x \sqrt{h} \epsilon_{ab} \tilde{Q}^{ab}, \quad (10)$$

<sup>6</sup>Note that the Zeroth Law, *i.e.*, the constancy of the surface gravity over a stationary event horizon, follows if the latter is also a Killing horizon[82, 83]. This is significant since the Zeroth Law is actually an essential ingredient to the entire framework of black hole thermodynamics.

yields the correct value for  $\mathcal{S}$  when evaluated not only at the bifurcation surface, but in fact on an arbitrary cross-section of the Killing horizon[74, 75].

Using Wald's technique, the formula for black hole entropy has been found for a general Lagrangian of the following form:

$$L = L(g_{ab}, R_{abcd}, \nabla_e R_{abcd}, \nabla_{(e_1} \nabla_{e_2)} R_{abcd}, \dots; \psi, \nabla_a \psi, \nabla_{(a_1} \nabla_{a_2)} \psi, \dots) ,$$

involving the Riemann tensor and symmetric derivatives of  $R_{abcd}$  (and the matter fields, denoted by  $\psi$ ) up to some finite order  $n$ .  $\mathcal{S}$  may then be written[73]–[75]

$$\mathcal{S} = -2\pi \oint d^2x \sqrt{h} \sum_{m=0}^n (-)^m \nabla_{(e_1} \dots \nabla_{e_m)} Z^{e_1 \dots e_m : abcd} \epsilon_{ab} \epsilon_{cd} \quad (11)$$

where the  $Z$ -tensors are defined by

$$Z^{e_1 \dots e_m : abcd} \equiv \frac{\delta L}{\delta \nabla_{(e_1} \dots \nabla_{e_m)} R_{abcd}} . \quad (12)$$

As a more explicit example, consider a polynomial-in- $R$  action

$$I = \frac{1}{16\pi G} \int d^4x \sqrt{-g} (R + a_2 R^2 + a_3 R^3) \quad (13)$$

for which one finds the simple result

$$\mathcal{S} = \frac{1}{4G} \oint d^D x \sqrt{h} (1 + 2a_2 R + 3a_3 R^2) . \quad (14)$$

Similarly in the Gauss-Bonnet theory with action

$$I = \frac{1}{16\pi G} \int d^4x \sqrt{-g} (R + \alpha(R_{abcd} R^{abcd} - 4R_{ab} R^{ab} + R^2)) \quad (15)$$

one finds from eq. (11) that

$$\mathcal{S} = \frac{1}{4G} \oint d^D x \sqrt{h} (1 + 2\alpha \tilde{R}(h)) . \quad (16)$$

where  $\tilde{R}(h)$  is the Ricci scalar calculated for the induced metric  $h_{ab}$ . In both eqs. (14) and (16), the first term yields the expected contribution for Einstein gravity, namely  $A/(4G)$ . Thus just as the Einstein term in the action is corrected by higher-curvature terms, the Einstein contribution to the black hole entropy receives higher-curvature corrections.

#### 4. Epilogue

While higher curvature theories are typically pathological when considered as fundamental, they may still be studied within the framework of effective field theory[6] where they produce minor corrections to Einstein gravity. Still perturbative investigations[28]–[33] of black holes in this context have revealed modifications to black hole thermodynamics and the generation of new scalar hair in these theories. It may be of interest to the perturbative framework on a more formal footing, *e.g.*, addressing questions such as whether or not solutions to eq. (2) always exist which leave the event horizon a regular surface.

The absence of ghosts in the Lovelock, dilatonic Gauss-Bonnet and polynomial-in- $R$  theories also seems an interesting question to investigate more fully. This feature seems to make these theories an interesting mathematical framework in which to study exact black hole solutions of the full higher derivative equations. It may be of interest then to find more general stationary solutions, *i.e.*, rotating black hole solutions. The stability of the event horizon, however, remains an important question to be addressed for even the solutions with spherical symmetry.

Ultimately it was the studies of various black holes solutions that led to the beautiful generalization of black hole thermodynamics for higher curvature theories. Wald’s new derivation of the First Law[72] demonstrates the black hole entropy is always determined by a local geometric expression evaluated at the event horizon, and provides a general and explicit formula (11) for  $\mathcal{S}$ . It may be that these results may be used to provide an even more refined test of our understanding of black hole entropy in superstring theory, given the recent dramatic progress in that area[9]. One interesting question related to Wald’s derivation is determining the minimal requirements for a stationary event horizon to be a Killing horizon.

While eq. (11) provides an elegant expression for  $\mathcal{S}$ , it should be noted that amongst the details overlooked in sect. 3.1 was the fact that a number of ambiguities arise in the construction of  $\tilde{Q}^{ab}$  [74, 75]. Hence eq. (11) should be understood as the result of making certain (natural) choices in the calculation. None of these ambiguities have any effect when  $\mathcal{S}$  is evaluated on a stationary horizon, but they might become significant for non-stationary horizons. As an example, it should be noted that for the Gauss-Bonnet theory, eqs. (11) and (16) are strictly not the same because the latter expression relies on the additional information that the extrinsic curvature of the bifurcation surface vanishes[70]. Hence the two expressions should not be expected to coincide on a non-stationary horizon.

An important guide to resolving these ambiguities should be the Second Law. Certainly if  $\mathcal{S}$  is to play the role of an entropy, it should also satisfy

the Second Law of black hole thermodynamics as a black hole evolves, *i.e.*, there should be a classical increase theorem for any dynamical processes. Within Einstein gravity, the Second Law is established by Hawking's area theorem[84]. So far only limited results have been produced for higher curvature theories[85] — see also comments in ref. [78]. One can show for quasistationary processes that the Second Law is in fact a direct consequence of the First Law (and a local positive energy condition for the matter fields), independent of the details of the gravitational dynamics. For polynomial-in- $R$  theories, one can establish the Second Law with certain restrictions on the coupling constants (and again a positive energy condition on the matter sector). One proof of the latter involves studying the properties of the null rays along the event horizon with an extension of the Raychaudhuri equation. A valuable extension of these results[85] would be to establish the Second Law within a perturbative framework. Of course, another important question is to determine the validity of the generalized Second Law[86] for evaporating black holes in the higher curvature theories.

A final question which I pose here is related to the conformal anomaly[87]. In quantum field theory[7], the conformal anomaly may be characterized as higher derivative modifications of the renormalized gravitational equations of motion which do not result from the variation of a local action. It appears that these terms will modify the expression for the black hole entropy[88] but they can not in general be addressed with Wald's construction — see, however, [89]. It would be interesting to improve on the latter to systematically include the effects of these contributions, and to determine whether  $\mathcal{S}$  or its variation in the First Law still retains its local geometric character. The investigation of a two-dimensional toy model would seem to indicate that the answer to the last question is negative[89].

I would like to gratefully acknowledge useful discussions with T. Jacobson. I am also grateful for the hospitality of the Institute for Theoretical Physics at the University of California, Santa Barbara, where this manuscript was completed. This work was supported by NSERC of Canada, and at the ITP, UCSB by NSF Grant PHY94-07194.

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